Mediatic Graphs*

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Abstract

A medium is a type of semigroup on a set of states, constrained by strong axioms. Any medium can be represented as an isometric subgraph of the hypercube, with each token of the medium represented by a particular equivalence class of arcs of the subgraph. Such a representation, although useful, is not especially revealing of the structure of a particular medium. We propose an axiomatic definition of the concept of a 'mediatic graph'. We prove that the graph of any medium is a mediatic graph. We also show that, for any non-necessarily finite set $\mathcal{S}$, there exists a bijection from the collection $\mathfrak{M}$ of all the media on $\mathcal{S}$ of states onto the collection $\mathfrak{G}$ of all the mediatic graphs on $\mathcal{S}$. A change of framework for media is noteworthy: the concept of a medium is specified here in terms of two axioms, rather than the original four.

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Background and Introduction

The core concept of this paper can occur in the guise of various representations. Four of them are relevant here, the last one being new.

1. A MEDIUM, that is, a semigroup of transformations on a set of states, constrained by strong axioms (see Falmagne, 1997; Falmagne and Ovchinnikov, 2002).

2. An ISOMETRIC SUBGRAPH OF THE HYPERCUBE, OR "PARTIAL CUBE." By "isometric", we mean that the distance between any two vertices of the subgraph is identical to the distance between the same two vertices in the hypercube (Graham and Pollak, 1971; Djoković, 1973). Each state of the medium is mapped to a vertex of the graph, and each transformation corresponds to an equivalence class of its arcs. Note that, as will become clear later on, no assumption of finiteness is made in this or in any of the other representation.

3. An ISOMETRIC SUBGRAPH OF THE INTEGER LATTICE. This representation is not exactly interchangeable with the preceding one. While it is true that any isometric subgraph of the hypercube is representable as an isometric subgraph of the integer lattice and vice versa, the latter representation lands in a space equipped with a considerable amount of structure. Notions of 'lines', 'hyperplanes', or 'parallelism' can be legitimately defined if one wishes. Moreover, the dimension of the lattice representation is typically much smaller than that of the partial cube representing the same medium and so can be valuable in the representation of large media (see, in particular, Eppstein, 2005, in which an algorithm is described for finding the minimum dimension of a lattice representation of a partial cube).

4. A MEDIATIC GRAPH. Axiomatic definitions are usually regarded as preferable whenever feasible, and that is what is given here.

The definition of a medium is recalled in the next section, together with some key concepts and the consequences of the axioms that are useful for this paper. Note that two axioms are used, which are equivalent to the original four used by Falmagne (1997) (see also Falmagne and Ovchinnikov, 2002; Eppstein, 2002). The graph of a medium and those graphs that induce media, called 'mediatic graphs' are defined and studied in the following two sections. The last two sections of the paper are devoted to specifying the correspondence between mediatic graphs and media, for a given possibly infinity set—of vertices or states depending on the case.

The subject of this paper may at first seem to be singularly ill chosen for a volume honoring Peter Fishburn's, as its topic does not readily evoke any of Peter's favorite concepts. But the enormously rich span of his accomplishment is not so easily escaped: indeed, the set of all interval orders (Fishburn, 1971) on any finite set is representable as a mediatic graph, and so is the set of all semiorders (Fishburn, 1985; Fishburn and Trotter, 1999) on the same set, these three citations heading a list far too long to be included here[1]. For the representability of families of interval orders or semiorders by mediatic graphs, see the concluding paragraph of this paper

[1] In view of the constraints set by the editors of this volume on the length of the many contributing papers.

The Concept of a Medium

We begin with the terminology of 'token systems' which provides a convenient framework.

1 Definition. Let $\mathcal{S}$ be a set of *states*. A *token* is a function $\tau : S \mapsto S\tau$ mapping $\mathcal{S}$ into itself. We shall use the abbreviations $S\tau = \tau(S)$, and $S\tau_1\tau_2 \cdots \tau_n = \tau_n[\cdots \tau_2[\tau_1(S)] \cdots]$ for the function composition. By definition, the identity function τ_0 on $\mathcal{S}$ is not a token. Let $\mathcal{T}$ be a set of tokens on $\mathcal{S}$. The pair $(\mathcal{S}, \mathcal{T})$ is called a *token system*. We suppose that $|\mathcal{S}| \geq 2$ and $\mathcal{T} \neq \varnothing$.

Let V and S be two distinct states. Then V is *adjacent* to S if $S\tau = V$ for some token τ. A token $\tilde{\tau}$ is a *reverse* of a token τ if, for any two adjacent states S and V, we have

$$S\tau = V \quad \Longleftrightarrow \quad V\tilde{\tau} = S, \tag{1}$$

and thus $S\tau\tilde{\tau} = S$. It is straightforward that a token has at most one reverse. If the reverse $\tilde{\tau}$ of a token τ exists, then $\tilde{\tilde{\tau}} = \tau$; that is, τ and $\tilde{\tau}$ are mutual reverses. If every token has a reverse, then adjacency is a symmetric relation on $\mathcal{S}$.

2 Definition. A *message* is a string of elements of the set of tokens $\mathcal{T}$. The message $\tau_1 \ldots \tau_n$ defines a function $S \mapsto S\tau_1 \cdots \tau_n$ on the set of states $\mathcal{S}$. If $\boldsymbol{m} = \tau_1 \ldots \tau_n$ denotes a message, we also (by abuse of notation) write $\boldsymbol{m} = \tau_1 \cdots \tau_n$ for the corresponding function. No ambiguity will arise from this double usage.

A message may consist in (the symbol representing) a single token. The *content* of a message $\boldsymbol{m} = \tau_1 \ldots \tau_n$ is the set $\mathcal{C}(\boldsymbol{m}) = \{\tau_1, \ldots, \tau_n\}$ of its tokens. We write $\ell(\boldsymbol{m}) = n$ to denote the *length* of the message $\boldsymbol{m}$. (We have thus $|\mathcal{C}(\boldsymbol{m})| \leq \ell(\boldsymbol{m})$.) A message $\boldsymbol{m}$ is *effective* (resp. *ineffective*) for a state S if $S\boldsymbol{m} \neq S$ (resp. $S\boldsymbol{m} = S$) for the function $S \mapsto S\boldsymbol{m}$. A message $\boldsymbol{m} = \tau_1 \ldots \tau_n$ is *stepwise effective* for S if $S\tau_1 \cdots \tau_k \neq S\tau_0 \cdots \tau_{k-1}$, $1 \leq k \leq n$. A message which is both stepwise effective and ineffective for some state is called a *return message* or, more brief, a *return* (for that state).

A message is *consistent* if it does not contain both a token and its reverse, and *inconsistent* otherwise. Two messages $\boldsymbol{m}$ and $\boldsymbol{n}$ are *jointly consistent* if $\boldsymbol{mn}$ (or, equivalently, $\boldsymbol{nm}$) is consistent. A consistent message which is stepwise effective for some state S and does not have any of its token occurring more than once is said to be *concise* (for S). A message $\boldsymbol{m} = \tau_1 \ldots \tau_n$ is *vacuous* if the set of indices $\{1, \ldots, n\}$ can be partitioned into pairs $\{i, j\}$, such that τ_i and τ_j are mutual reverses. By abuse of language, we sometimes call 'empty' a place holder symbol that can be deleted, as in: 'let $\boldsymbol{mn}$ be a message in which $\boldsymbol{n}$ is either a concise message or is empty' (that is $\boldsymbol{mn} = \boldsymbol{m}$). If $\boldsymbol{m} = \tau_1 \ldots \tau_n$ is a stepwise effective message producing a state V from a state S, then the *reverse* of $\boldsymbol{m}$ is defined by $\widetilde{\boldsymbol{m}} = \tilde{\tau}_n \ldots \tilde{\tau}_1$. We then have clearly $V\widetilde{\boldsymbol{m}} = S$ and moreover $\tau \in \mathcal{C}(\boldsymbol{m})$ if and only if $\tilde{\tau} \in \mathcal{C}(\widetilde{\boldsymbol{m}})$.

3 Axioms for Medium. A token system $(\mathcal{S}, \mathcal{T})$ is called a *medium (on $\mathcal{S}$)* if the two following axioms are satisfied.

[Ma] For any two distinct states S, V in $\mathcal{S}$, there is a concise message producing V from S.

[Mb] Any return message is vacuous.

A medium $(\mathcal{S}, \mathcal{T})$ is *finite* if $\mathcal{S}$ is a finite set. The concept of a medium was proposed by Falmagne (1997) who proved various basic facts about media. Other results were obtained

by Falmagne and Ovchinnikov (2002) (see also Ovchinnikov and Dukhovny, 2000; Eppstein and Falmagne, 2002; Ovchinnikov, 2006).

Four different axioms[2] were used in the papers cited above to define the concept of a medium, which are equivalent to Axioms [Ma] and [Mb]. Specifically, we have the following result:

4 Theorem. *A token system $(\mathcal{S}, \mathcal{T})$ is a medium if and only if the following four conditions hold:*

[M1] *Any token has a reverse.*

[M2] *For any two distinct states S, V in $\mathcal{S}$ there is a consistent message transforming S into V.*

[M3] *A message which is stepwise effective for some state is ineffective for that state if and only if it is vacuous.*

[M4] *Two stepwise effective, consistent messages producing the same state are jointly consistent.*

We omit the simple proof of the equivalence between [Ma]-[Mb] and [M1]-[M4].

Some Basic Results

The material in this section, only part of which is new, is instrumental for the graph-theoretical results presented in this paper. We omit the proofs of previously published results (see Falmagne, 1997; Falmagne and Ovchinnikov, 2002).

5 Lemma. (i) *No token can be identical to its own reverse.*

(ii) *Let $\boldsymbol{m}$ be a message that is concise for some state; we have then $l(m) = |C(m)|$ and $\mathcal{C}(\boldsymbol{m}) \cap \mathcal{C}(\widetilde{\boldsymbol{m}}) = \varnothing$.*

(iii) *For any two adjacent (thus, distinct) states S and V, there is exactly one token producing V from S.*

(iv) *No token can be a 1-1 function.*

(v) *Suppose that $\boldsymbol{m}$ and $\boldsymbol{n}$ are stepwise effective for S and V, respectively, with $S\boldsymbol{m} = V$ and $V\boldsymbol{n} = W$. Then $\boldsymbol{mn}$ is stepwise effective for S, with $S\boldsymbol{mn} = W$.*

(vi) *Let $\boldsymbol{m}$ and $\boldsymbol{n}$ be two distinct concise messages transforming some state S. Then*

$$S\boldsymbol{m} = S\boldsymbol{n} \quad \Longleftrightarrow \quad \mathcal{C}(\boldsymbol{m}) = \mathcal{C}(\boldsymbol{n}).$$

Lemma 5(vi) suggests an important concept.

6 Definition. Let $(\mathcal{S}, \mathcal{T})$ be a medium. For any state S, define the *(token) content* of S as the set $\widehat{S}$ of all tokens each of which is contained in at least one concise message producing S; formally:

$$\widehat{S} = \{\tau \in \mathcal{T} \,|\, \exists V \in \mathcal{S}, V\boldsymbol{m} = S, \text{ for } \boldsymbol{m} \text{ concise with } \tau \in \mathcal{C}(\boldsymbol{m})\}.$$

We refer to the family $\widehat{\mathcal{S}}$ of all the contents of the states in $\mathcal{S}$ as the *content family* of the medium $(\mathcal{S}, \mathcal{T})$.

[2] Falmagne (1997) used a slightly different definition of 'reverse', allowing the possibility of several reverses for a given token. This was compensated by a stronger version of [M1] requiring the existence of a unique reverse for every token.

7 Remark. Because any two stepwise effective, consistent messages producing the same state must be jointly consistent (Condition [M4] in Theorem 4), the content of a state cannot contain both a token and its reverse.

Writing $\triangle$ for the symmetric set difference, and $+$ for the disjoint union, we have:

8 Theorem. *If $S\boldsymbol{m} = V$ for some concise message $\boldsymbol{m}$ (thus $S \neq V$), then $\widehat{V} \setminus \widehat{S} = \mathcal{C}(\boldsymbol{m})$, and so $\widehat{V} \triangle \widehat{S} = \mathcal{C}(\boldsymbol{m}) + \mathcal{C}(\widetilde{\boldsymbol{m}})$.*

9 Theorem. *For any token τ and any state S, we have either $\tau \in \widehat{S}$ or $\tilde{\tau} \in \widehat{S}$; so, $|\widehat{S}| = |\widehat{V}|$ for any two states S and V with $S = V$ if and only if $\widehat{S} = \widehat{V}$. Moreover, if $\mathcal{S}$ is finite, then $|\widehat{S}| = |\mathcal{T}|/2$ for any $S \in \mathcal{S}$.*

10 Definition. If $\boldsymbol{m}$ and $\boldsymbol{n}$ are two concise messages producing, from a state S, the same state $V \neq S$, we call $\boldsymbol{m}\widetilde{\boldsymbol{n}}$ an *orderly circuit* for S.

By Axiom [Mb], an orderly circuit is vacuous; therefore its length must be even. The following result is of general interest for orderly circuits.

11 Theorem. *Let S, N, Q and W be four distinct states of a medium and suppose that*

$$N\tau = S, \quad W\mu = Q, \quad S\boldsymbol{q} = N\boldsymbol{q}' = Q, \quad S\boldsymbol{w}' = N\boldsymbol{w} = W \tag{2}$$

for some tokens τ and μ and some concise messages $\boldsymbol{q}$, $\boldsymbol{q}'$, $\boldsymbol{w}$ and $\boldsymbol{w}'$ (see Figure 1). Then, the four following conditions are equivalent:

(i) $\ell(\boldsymbol{q}) + \ell(\boldsymbol{w}) \neq \ell(\boldsymbol{q}') + \ell(\boldsymbol{w}')$ *and* $\mu \neq \tilde{\tau}$;

(ii) $\tau = \mu$;

(iii) $\mathcal{C}(\boldsymbol{q}) = \mathcal{C}(\boldsymbol{w})$ *and* $\ell(\boldsymbol{q}) = \ell(\boldsymbol{w})$;

(iv) $\ell(\boldsymbol{q}) + \ell(\boldsymbol{w}) + 2 = \ell(\boldsymbol{q}') + \ell(\boldsymbol{w}')$.

Moreover, any of these conditions implies that $\boldsymbol{q}\tilde{\mu}\widetilde{\boldsymbol{w}}\tau$ is an orderly circuit for S with $S\boldsymbol{q}\tilde{\mu} = S\tilde{\tau}\boldsymbol{w} = W$. The converse does not hold.

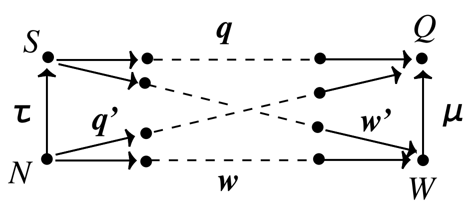

Figure 1: For Theorem 11. Illustration of the conditions listed in (2).

Proof. We prove (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Suppose that $\tau \neq \mu$. The token $\tilde{\tau}$ must occur exactly once in either $\boldsymbol{q}$ or in $\widetilde{\boldsymbol{w}}$. Indeed, we have $\mu \neq \tilde{\tau}$, both $\boldsymbol{q}$ and $\boldsymbol{w}$ are concise, and the message $\tau\boldsymbol{q}\tilde{\mu}\widetilde{\boldsymbol{w}}$ is a return for S, and so is vacuous by [Ma]. It can be verified that each of the two mutually exclusive, exhaustive cases: [a] $\tilde{\tau} \in \mathcal{C}(\boldsymbol{q}) \cap \mathcal{C}(\boldsymbol{w}')$; and [b] $\tilde{\tau} \in \mathcal{C}(\widetilde{\boldsymbol{w}}) \cap \mathcal{C}(\widetilde{\boldsymbol{q}}')$ lead to

$$\ell(\boldsymbol{q}) + \ell(\boldsymbol{w}) = \ell(\boldsymbol{q}') + \ell(\boldsymbol{w}'), \tag{3}$$

contradicting (i). Thus, we must have $\tau = \mu$.

We only prove Case [a]. The other case is treated similarly. Since $\tilde{\tau}$ is in $\mathcal{C}(\boldsymbol{q})$, neither τ nor $\tilde{\tau}$ can be in $\mathcal{C}(\boldsymbol{q}')$. Indeed, both $\boldsymbol{q}$ and $\boldsymbol{q}'$ are concise and $\boldsymbol{q}\widetilde{\boldsymbol{q}}'\tau$ is a return for S. It follows that both $\tilde{\tau}\boldsymbol{q}'$ and $\boldsymbol{q}$ are concise messages producing Q from S. By Theorem 8, we must have $\mathcal{C}(\tilde{\tau}\boldsymbol{q}') = \mathcal{C}(\boldsymbol{q})$, which implies $\ell(\tilde{\tau}\boldsymbol{q}') = \ell(\boldsymbol{q})$, and so

$$\ell(\boldsymbol{q}) = \ell(\boldsymbol{q}') + 1. \tag{4}$$

A argument along the same lines shows that

$$\ell(\boldsymbol{w}) + 1 = \ell(\boldsymbol{w}'). \tag{5}$$

Adding (4) and (5) and simplifying, we obtain (3). The proof of Case [b] is similar.

(ii) $\Leftrightarrow$ (iii). If $\mu = \tau$, it readily follows (since both $\boldsymbol{q}$ and $\boldsymbol{w}$ are concise and $S\boldsymbol{q}\tilde{\tau}\widetilde{\boldsymbol{w}}\tau = S$) that any token in $\boldsymbol{q}$ must have a reverse in $\widetilde{\boldsymbol{w}}$ and vice versa. This implies $\mathcal{C}(\boldsymbol{q}) = \mathcal{C}(\boldsymbol{w})$, which in turn imply $\ell(\boldsymbol{q}) = \ell(\boldsymbol{w})$, and so (iii) holds. As $\boldsymbol{q}\tilde{\mu}\widetilde{\boldsymbol{w}}\tau$ is vacuous, it is clear that (iii) implies (ii).

(iii) $\Rightarrow$ (iv). Since (iii) implies (ii), we have $\tau \in \widehat{Q}\setminus\widehat{N}$ by Theorem 8. But both $\boldsymbol{q}$ and $\boldsymbol{q}'$ are concise, so $\tau \in \mathcal{C}(\boldsymbol{q}')\setminus\mathcal{C}(\boldsymbol{q})$. As $\tau\boldsymbol{q}\widetilde{\boldsymbol{q}}'$ is vacuous for N, we must have $\mathcal{C}(\boldsymbol{q})+\{\tau\} = \mathcal{C}(\boldsymbol{q}')$, yielding

$$\ell(\boldsymbol{q}) + 1 = \ell(\boldsymbol{q}'). \tag{6}$$

A similar argument gives $\mathcal{C}(\boldsymbol{w}) + \{\tau\} = \mathcal{C}(\boldsymbol{w}')$ and

$$\ell(\boldsymbol{w}) + 1 = \ell(\boldsymbol{w}'). \tag{7}$$

Adding (6) and (7) yields (iv).

(iv) $\Rightarrow$ (i). As (iv) is a special case of the first statement in (i), we only have to prove that $\mu \neq \tilde{\tau}$. Suppose that $\mu = \tilde{\tau}$. We must assign the token $\tilde{\tau}$ consistently so to ensure the vacuousness of the messages $\boldsymbol{q}\widetilde{\boldsymbol{q}}'\tau$ and $\tau\boldsymbol{w}'\widetilde{\boldsymbol{w}}$. By Theorem 8, $\mathcal{C}(\boldsymbol{q}) = \widehat{Q} \setminus \widehat{S}$. Since $\tilde{\tau} \in \widehat{Q}$ and, by Theorem 9, $\tilde{\tau} \notin \widehat{S}$, the only possibility is $\tilde{\tau} \in \mathcal{C}(\boldsymbol{q}) \setminus \mathcal{C}(\boldsymbol{q}')$. For similar reasons $\tau \in \mathcal{C}(\boldsymbol{w}) \setminus \mathcal{C}(\boldsymbol{w}')$. We obtain the two concise messages $\tilde{\tau}\boldsymbol{q}'$ and $\boldsymbol{q}$ producing Q from S, and the two concise messages $\boldsymbol{w}$ and $\tau\boldsymbol{w}'$ producing W from N. This gives $\ell(\boldsymbol{q}) = \ell(\tilde{\tau}\boldsymbol{q}')$ and $\ell(\boldsymbol{w}) = \ell(\tau\boldsymbol{w}')$. We obtain so $\ell(\boldsymbol{q}) = \ell(\boldsymbol{q}') + 1$ and $\ell(\boldsymbol{w}) = \ell(\boldsymbol{w}') + 1$, which leads to $\ell(\boldsymbol{q}) + \ell(\boldsymbol{w}) = \ell(\boldsymbol{q}') + \ell(\boldsymbol{w}') + 2$ and contradicts (iv). Thus, (iv) implies (i). We conclude that the four conditions (i)-(iv) are equivalent.

We now show that, under the hypotheses of the theorem, (ii) implies that $\boldsymbol{q}\tilde{\mu}\widetilde{\boldsymbol{w}}\tau$ is an orderly return for S with $S\boldsymbol{q}\tilde{\mu} = S\tilde{\tau}\boldsymbol{w} = W$. Both $\boldsymbol{q}$ and $\boldsymbol{w}$ are concise by hypothesis. We cannot have μ in $\mathcal{C}(\boldsymbol{q})$ because then $\tilde{\mu}$ is in $\mathcal{C}(\widetilde{\boldsymbol{q}})$ and the two concise messages $\widetilde{\boldsymbol{q}}$ and $\tau = \mu$ producing S are not jointly consistent, yielding a contradiction of Condition [M4] in Theorem 4. Similarly, we cannot have $\tilde{\mu}$ in $\mathcal{C}(\boldsymbol{q})$ since the two concise messages $\boldsymbol{q}$ and μ producing Q would not be jointly consistent. Thus, $\boldsymbol{q}\tilde{\mu}$ is a concise message producing W from S. For like reasons, with $\tau = \mu$, $\tilde{\tau}\boldsymbol{w}$ is a concise message producing W from S. We conclude that, with $\tau = \mu$, the message $\boldsymbol{q}\tilde{\mu}\widetilde{\boldsymbol{w}}\tau$ is an orderly return for S. The example of Figure 2, in which we have

$$\mu \neq \tau, \quad \boldsymbol{q} = \alpha\tilde{\tau}, \quad \boldsymbol{w} = \tilde{\mu}\alpha, \quad \boldsymbol{w}' = \alpha\tilde{\tau}\tilde{\mu}, \text{ and } \boldsymbol{q}' = \alpha,$$

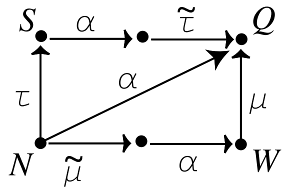

Figure 2: Under the hypotheses of Theorem 11, the hypothesis that $\boldsymbol{q}\tilde{\mu}\widetilde{\boldsymbol{w}}\tau$ is an orderly circuit for S does not imply $\tau = \mu$, with $\boldsymbol{q} = \alpha\tilde{\tau}$, $\boldsymbol{w} = \tilde{\mu}\alpha$, $\boldsymbol{q}' = \alpha$, and $\boldsymbol{w}' = \alpha\tilde{\tau}\tilde{\mu}$.

displays the orderly return $\alpha\tilde{\tau}\tilde{\mu}\tilde{\alpha}\mu\tau$ for S. It serves as a counterexample to the implication: if $\boldsymbol{q}\tilde{\mu}\widetilde{\boldsymbol{w}}\tau$ is an orderly return for S, then $\tau = \mu$. $\square$

In Definition 10, the concept of an orderly circuit was specified with respect to a particular state. The next definition and theorem concern a situation in which a circuit is orderly with respect to everyone of its states. In such a case, any token occurring in the circuit must have its reverse at the exact 'opposite' place in the circuit (see Theorem 13(i)).

12 Definition. Let $\tau_1 \dots \tau_{2n}$ be an orderly return for a state S. For $1 \leq i \leq n$, the two tokens τ_i and τ_{i+n} are called *opposite*. A return $\tau_1 \dots \tau_{2n}$ from S is *regular* if it is orderly and, for $1 \leq i \leq n$, the message $\tau_i\tau_{i+1} \dots \tau_{i+n-1}$ is concise for $S\tau_1 \cdots \tau_{i-1}$.

13 Theorem. *Let $\boldsymbol{m} = \tau_1 \dots \tau_{2n}$ be an orderly return for some state S. Then the following three conditions are equivalent.*

(i) *The opposite tokens of $\boldsymbol{m}$ are mutual reverses.*

(ii) *The return $\boldsymbol{m}$ is regular.*

(iii) *For $1 \leq i \leq 2n-1$, the message $\tau_i \dots \tau_{2n} \dots \tau_{i-1}$ is an orderly return for the state $S\tau_1 \cdots \tau_{i-1}$.*

Proof. We prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). In what follows $S_i = S\tau_0\tau_1 \dots \tau_i$ for $0 \leq i \leq 2n$, so $S_0 = S_{2n} = S$.

(i) $\Rightarrow$ (ii). Since $\boldsymbol{m}$ is an orderly return, for $1 \leq j \leq n$, there is only one occurrence of the pair $\{\tau_j, \tilde{\tau}_j\}$ in $\boldsymbol{m}$. Since $\tilde{\tau}_j = \tau_{j+n}$, there are no occurrences of $\{\tau_j, \tilde{\tau}_j\}$ in $\boldsymbol{p} = \tau_i \cdots \tau_{i+n-1}$, so it is a concise message for S_{i-1}.

(ii) $\Rightarrow$ (iii). Since $\boldsymbol{m}$ is a regular return, any message $\boldsymbol{p} = \tau_i \cdots \tau_{i+n-1}$ is concise, so any token of this message has a reverse in the message $\boldsymbol{q} = \tau_{i+n} \dots \tau_{2n} \dots \tau_{i-1}$. Since $\boldsymbol{p}$ is concise and $\ell(\boldsymbol{q}) = n$, the message $\boldsymbol{q}$ is concise. It follows that $\boldsymbol{pq}$ is an orderly return for the state S_{i-1}.

(iii) $\Rightarrow$ (i). Since the message $\tau_i \dots \tau_{2n} \dots \tau_{i-1}$ is an orderly return for S_{i-1}, the messages $\boldsymbol{q} = \tau_{i+1} \dots \tau_{i+n-1}$ and $\boldsymbol{q}' = \tau_i \dots \tau_{i+n-1}$ are concise for the states $S' = S_i$ and $N = S_{i-1}$, respectively, and produce the state $Q = S_{i+n-1}$. Likewise, the messages $\boldsymbol{w} = \tilde{\tau}_{i-1} \dots \tilde{\tau}_{2n} \dots \tilde{\tau}_{i+n}$ and $\boldsymbol{w}' = \tilde{\tau}_i \dots \tilde{\tau}_{2n} \dots \tilde{\tau}_{i+n}$ are concise for the states $N = S_{i-1}$ and $S' = S_i$, respectively, and produce the state $W = S_{i+n}$. It is clear that $\ell(\boldsymbol{q}) + \ell(\boldsymbol{w}) + 2 = \ell(\boldsymbol{q}') + \ell(\boldsymbol{w}')$. By Theorem 11, $\tau_{i+n} = \tilde{\tau}_i$. $\square$

The Graph of a Medium

For graph-theoretical concepts and terminology, we usually follow Bondy (1995).

14 Definition. A *graph representation* of a medium $(\mathcal{S}, \mathcal{T})$ is a bijection $\gamma : \mathcal{S} \to V$, where V is a set of vertices of a graph (V, E), such that two distinct states S and T are adjacent whenever $\{\gamma(S), \gamma(T)\}$ is an edge of the graph; formally,

$$\{\gamma(S), \gamma(T)\} \in E \quad \Longleftrightarrow \quad (\exists \tau \in \mathcal{T})(S\tau = T) \quad (S, T \in \mathcal{S},\ S \neq T). \tag{8}$$

We say then that the graph (V, E), which has no loops, represents the medium. A graph (V, E) representing a medium $(\mathcal{S}, \mathcal{T})$ is called the *graph of the medium* $(\mathcal{S}, \mathcal{T})$ if $V = \mathcal{S}$, the edges in E are defined as in (8), and γ is the identity mapping. Clearly, any medium has its graph. We shall prove in this paper that the converse also holds, namely: the graph of a medium defines its medium (see Theorem 36). We recall that two graphs (V, E) and (V', E') are isomorphic if there is a bijection $\varphi : V \to V'$ such that

$$\{P, Q\} \in E \Longleftrightarrow \{\varphi(P), \varphi(Q)\} \in E' \qquad (P, Q \in E,\ P \neq Q). \tag{9}$$

15 Lemma. *A graph isomorphic to a graph representing a medium $\mathcal{M}$ also represents $\mathcal{M}$.*

It is intuitively clear that shortest paths in the graph of a medium correspond to concise messages of that medium. Our next lemma states that fact precisely.

16 Lemma. *Let $\gamma : \mathcal{S} \to V$ be the representation of a medium $(\mathcal{S}, \mathcal{T})$ by a graph $G = (V, E)$. If $\boldsymbol{m} = \tau_1 \ldots \tau_m$ is a concise message producing a state T from a state S, then the sequence of vertices $(\gamma(S_i))_{0 \leq i \leq m}$, where $S_i = S\tau_0\tau_1 \cdots \tau_i$, for $0 \leq i \leq m$, forms a shortest path joining $\gamma(S)$ and $\gamma(T)$ in G. Conversely, if a sequence $(\gamma(S_i))_{0 \leq i \leq m}$ is a shortest path connecting $\gamma(S_0) = \gamma(S)$ and $\gamma(S_m) = \gamma(T)$, then $\boldsymbol{m} = \tau_1 \ldots \tau_m$ with $S\tau_0\tau_1 \cdots \tau_i = S_i$, for $0 \leq i \leq m$, is a concise message producing T from S.*

PROOF. (Necessity.) Let $\gamma(P_0) = \gamma(S), \gamma(P_1), \ldots, \gamma(P_n) = \gamma(T)$ be a path in G joining $\gamma(S)$ to $\gamma(T)$. Correspondingly, there is a stepwise effective message $\boldsymbol{n} = \rho_1 \cdots \rho_n$ such that $P_i = T\rho_1 \cdots \rho_{n-i}$ for $0 \leq i < n$. The message $\boldsymbol{mn}$ is a return for S. By Axiom [Mb], this message is vacuous. Since $\boldsymbol{m}$ is a concise message for S, we must have $\ell(\boldsymbol{m}) = m \leq \ell(\boldsymbol{n}) = n$.

(Sufficiency.) Let $\gamma(S_0) = \gamma(S), \gamma(S_1), \ldots, \gamma(S_m) = \gamma(T)$ be a shortest path from $\gamma(S)$ to $\gamma(T)$ in G. Then, there are some tokens τ_i, $1 \leq i \leq m$ such that $S_i\tau_{i+1} = S_{i+1}$ for $0 \leq i < m$. The message $\boldsymbol{m} = \tau_1 \ldots \tau_m$ produces the state T from the state S. An argument akin to that used in the foregoing paragraph shows that $\boldsymbol{m}$ is a concise message for S. □

We now establish a result of the same vein for the regular returns of a medium (cf. Definition 12).

17 Definition. We recall that a sequence of vertices $\boldsymbol{s}_m = (v_i)_{0 \leq i \leq m}$ such that $\{v_i, v_{i+1}\}$ are edges in a graph is a circuit if $v_m = v_0$ and all the vertices $v_1, \ldots, v_m$ are different. By abuse of language, we say that the edges $\{v_i, v_{i+1}\}$, for $0 \leq i \leq m-1$, *belong* to the circuit $\boldsymbol{s}_m$. The circuit $\boldsymbol{s}_m$ is *even* if it has an even number of edges: $m = 2n$; any two of its edges $\{v_i, v_{i+1}\}$ and $\{v_{i+n}, v_{i+n+1}\}$, $0 \leq i \leq n-1$ are then called *opposite*. A circuit is *minimal* if at least one shortest path between any two of its vertices is a segment of the circuit. A graph is *even* if all its circuits are even.

18 Lemma. *Let $\gamma : \mathcal{S} \to V$ be the representation of a medium $\mathcal{M} = (\mathcal{S}, \mathcal{T})$ by a graph $G = (V, E)$. If $\boldsymbol{m} = \tau_1 \dots \tau_{2n}$ is a regular return for some state $S \in \mathcal{S}$, then the sequence of vertices $(\gamma(S_i))_{0 \le i \le 2n}$, where $S_i = S\tau_0\tau_1 \cdots \tau_i$, for $0 \le i \le 2n$, forms an even, minimal circuit of G (with $S = S_0 = S_{2n}$). Conversely, if a sequence $(\gamma(S_i))_{0 \le i \le 2n}$ is an even minimal circuit of G, then $\boldsymbol{m} = \tau_1 \dots \tau_m$ with $S\tau_0\tau_1 \cdots \tau_i = S_i$, for $0 \le i \le 2n$ is a regular circuit for S in $\mathcal{M}$.*

PROOF. In the notation of the lemma, let $\boldsymbol{m}$ be a regular return for state S. Thus, by definition of a regular return (cf. 12), $\tau_1 \dots \tau_n$ and $\tilde{\tau}_{2n} \dots \tilde{\tau}_{n+1}$ are concise messages for S. By Lemma 16, the sequence of vertices $(\gamma(S_i))_{0 \le i \le n}$, where $S_i = S\tau_0\tau_1 \cdots \tau_i$, for $0 \le i \le n$, forms a shortest path joining $\gamma(S)$ and $\gamma(T)$, with $T = S\tau_1 \cdots \tau_n$. Similarly, the sequence $\gamma(S_{2n}), \gamma(S_{2n-1}), \dots, \gamma(S_{n+1})$ is another shortest path joining $\gamma(S)$ and $\gamma(T)$. Since γ is a 1-1 function, all the vertices $\gamma(S_i)$ are distinct, and so the sequence $(\gamma(S_i))_{0 \le i \le 2n}$ is an even circuit. This circuit is a minimal one. Indeed, by definition of a regular return, all the messages $\tau_i\tau_{i+1} \dots \tau_{i+n-1}$ are concise for $S\tau_1 \cdots \tau_{i-1}$. So, by Lemma 16, all the sequences $\gamma(S_i), \dots, \gamma(S_{i+n-1})$ are shortest paths between $\gamma(S_i)$ and $\gamma(S_{i+n-1})$, which implies that at least one shortest path between any two vertices of the circuit $(\gamma(S_i))_{0 \le i \le 2n}$ is a segment of that circuit. We omit the proof of the converse part of this lemma. The argument is based on the converse part of Lemma 16 and is similar. □

19 Remark. A close reading of this proof shows that opposite tokens τ_i, $\tau_{i+n} = \tilde{\tau}_i$ in a regular return correspond to opposite edges $\{\gamma(S_i), \gamma(S_{i+1})\}$, $\{\gamma(S_{i+n}, \gamma(S_{i+1+n})\}$ in the even minimal circuit of the representing graph, with $S_{i+1} = S_i\tau_i$ and $S_{i+n} = S_{i+n+1}\tau_{i+n}$.

Media Inducing Graphs

Our next task is to characterize the graphs representing media in terms of graph concepts. Some necessary conditions are easily inferred from the axioms of a medium. For example, Axiom [Ma] forces the graph to be connected, and [Mb] demands that it is even. By convention, the graph should not have any loops. However, as shown by the two example below, these three conditions are not sufficient to characterize the graph of a medium.

20 Two Counterexamples. The graphs corresponding to the digraphs A and B in Figure 3 are connected and all their circuits are even. Moreover, they have no loops. Yet, neither A nor B can yield the graph of a medium. We leave to the reader to prove this for Figure 3A.

Here is why in the case of B. The circuit pictured in thick lines is even and minimal. By Lemma 18, it must represent a regular return in a medium. From Remark 19, we know that the same token must be matched to opposite edges of the circuit. Accordingly, the same token ν has been assigned to the arcs JM and RW. (To simplify the figure, only one token from each pair of mutually reverse tokens is indicated.) The circuit containing the six vertices L, K, N, W, R and H is also even and minimal. Thus, the arcs LK and RW must be assigned the same token, and since RW has been assigned token ν, that token must also be assigned to LK. The argument governing the placement of the token τ are similar. The consequence, however, is that there is no concise message from L to J: any message producing J from L contains either both ν and $\tilde{\nu}$, or both $\tilde{\tau}$ and τ. This example will be crucial in our understanding of the appropriate axiomatization of a graph capable of representing a medium.

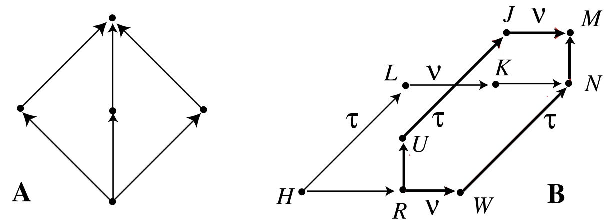

Figure 3: Neither of these graphs is that of a medium. The token system corresponding to Digraph **B** contradicts [Ma]. Which of the properties of a medium is contradicted by Digraph **A**?

In our failed attempt at representing a medium in Figure 3, we have chosen to picture the arcs representing the same token by parallel segments (forming two sides of an implicit rectangle). The intuition that the opposite arcs of even minimal circuits should be parallel is a sound one, and suggests the construction of an equivalence relation on the set of set of arcs of the digraph. Such a construction is delicate, however, and the two examples of media pictured below by their digraphs must be taken into account.

21 Examples. Together with the examples of Figure 3, Examples A and B in Figure 4 will also guide and illustrate our choice of concepts and axioms.

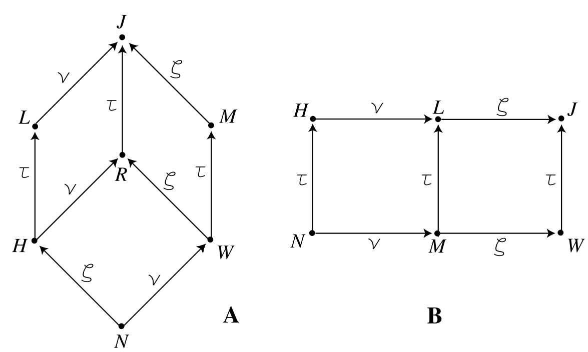

Figure 4: Two examples of graphs of media. In Example B, notice that different tokens are assigned to the opposite arcs HL and MW. This circuit is not minimal. Compare with the situation of the arcs LJ and NW in Example A.

22 Definition. We write $\vec{E} = \{ST \,|\, \{S,T\} \in E\}$ for the set of all the arcs of a graph $G = (V, E)$. The *like* relation of the graph G is a relation $\mathfrak{L}$ on $\vec{E}$ defined by

$$ST \,\mathfrak{L}\, PQ \iff (\delta(S,P)+1 = \delta(T,Q)+1 = \delta(S,Q) = \delta(T,P)) \qquad (\{S,T\},\ \{P,Q\} \in E),$$

where δ denotes the graph theoretical distance between the vertices of the graph. In Example B of Figure 4, we have $NH \,\mathfrak{L}\, WJ$ because

$$\delta(H,J) + 1 = \delta(N,W) + 1 = \delta(H,W) = \delta(N,J),$$

but $HL \,\mathfrak{L}\, MW$ does not hold since

$$\delta(H,M) = \delta(L,W) = 2 \quad \text{and} \quad 4 = \delta(H,W) \neq \delta(L,M) = 1.$$

The point is that the arcs HL and MW are opposite in the circuit H, L, J, W, M, N, H, but this circuit is not minimal.

The like relation is clearly reflexive and symmetric; and moreover

$$ST\,\mathfrak{L}\,PQ \quad \Longleftrightarrow \quad TS\,\mathfrak{L}\,QP \qquad (\{S,T\},\, \{P,Q\} \in E). \tag{10}$$

Two binary relations on the set of edges of a graph play a central role in characterizing partial cubes. They are Djoković's relation θ (Djoković, 1973) and Winkler's relation Θ (Winkler, 1984). These relations are germane to, but different from the like relation of this paper. Space limitation prevents us to specify the relationship here.

We now come to the main concept of this paper. We recall that a graph is bipartite if and only if it is even (König, 1916).

23 Definition. Let $G = (V, E)$ be a graph equipped with its like relation $\mathfrak{L}$. The graph G is called *mediatic* if the following three axioms hold.

[G1] G is connected.

[G2] G is bipartite.

[G3] $\mathfrak{L}$ is transitive.

The set of vertices is not assumed to be finite. It is easily verified that any graph isomorphic to a mediatic graph is mediatic.

Axiom [G3] eliminates the counterexample of Figure 3B. Indeed, since

$$\delta(L, J) = 4, \quad \delta(K, M) = 2, \quad \delta(L, M) = 3 = \delta(K, J)$$

we have

$$LK\,\mathfrak{L}\,RW\,\mathfrak{L}\,JM \quad \text{but not} \quad LK\,\mathfrak{L}\,JM.$$

The following result is immediate.

24 Lemma. *The like relation $\mathfrak{L}$ of a mediatic graph (V, E) is an equivalence relation on $\vec{E}$.*

25 Definition. We denote by

$$\langle ST \rangle = \{PQ \in \vec{E} \mid ST\,\mathfrak{L}\,PQ\}$$

the equivalence class containing the arc ST in the partition of $\vec{E}$ induced by $\mathfrak{L}$.

We will show that a graph representing a medium is mediatic (see Theorem 28). Our next lemma is the first step.

26 Lemma. *Let γ be the representation of a medium $\mathcal{M} = (\mathcal{S}, \mathcal{T})$ by a graph $G = (\mathcal{S}, E)$ which is equipped with its like relation $\mathfrak{L}$. Suppose that $\gamma(N)\gamma(S)\,\mathfrak{L}\,\gamma(W)\gamma(Q)$. Then $N\tau = S$ and $W\tau = Q$ for some $\tau \in \mathcal{T}$. In fact, there exists an orderly circuit $\boldsymbol{q}\tilde{\tau}\widetilde{\boldsymbol{w}}\tau$ for S in $\mathcal{M}$, with $S\boldsymbol{q}\tilde{\tau} = S\tilde{\tau}\boldsymbol{w} = W$; thus $\boldsymbol{q}$ and $\boldsymbol{w}$ are concise with $\ell(\boldsymbol{q}) = \ell(\boldsymbol{m})$. Such a circuit is not necessarily regular.*

Proof. We abbreviate our notation for this proof, and write $S^\gamma = \gamma(S)$ for all $S \in \mathcal{S}$. By definition, $N^\gamma S^\gamma \,\mathfrak{L}\, W^\gamma Q^\gamma$ implies that $\delta(S^\gamma, Q^\gamma) = \delta(N^\gamma, W^\gamma) = \delta(N^\gamma, Q^\gamma) - 1 = \delta(S^\gamma, W^\gamma) - 1$; so, there are, for some $n \in \mathbb{N}$, two shortest paths

$$S_0^\gamma = S^\gamma, S_1^\gamma, \ldots, S_n^\gamma = Q^\gamma \quad \text{and} \quad N_0^\gamma = N^\gamma, N_1^\gamma, \ldots, N_n^\gamma = W^\gamma$$

between S^γ and Q^γ, and N^γ and W^γ, respectively. Moreover,

$$S_0^\gamma = S^\gamma, S_1^\gamma, \ldots, S_n^\gamma = Q^\gamma, W^\gamma \text{ and } N_0^\gamma = N^\gamma, N_1^\gamma, \ldots, N_n^\gamma = W^\gamma, Q^\gamma$$

are also shortest paths. Using Lemma 16, we can assert the existence of two concise messages $\boldsymbol{q}$ and $\boldsymbol{w}$ such that $S\boldsymbol{q} = Q$ and $N\boldsymbol{w} = W$, with $\ell(\boldsymbol{q}) = \ell(\boldsymbol{w}) = n$. Also, for some tokens τ and μ, we have $N\tau = S$ and $W\mu = Q$ with $\boldsymbol{q}' = \tau\boldsymbol{q}$ and $\boldsymbol{w}' = \tilde{\tau}\boldsymbol{w}$ concise for N and S, respectively, and $\ell(\boldsymbol{q}') = \ell(\boldsymbol{w}') = n+1$. We are exactly in the situation of Theorem 11 (see Figure 1). Using the implication (iv) $\Rightarrow$ (ii) of this theorem, we obtain $\tau = \mu$. Condition (iv) also implies that $\boldsymbol{q}\tilde{\tau}\tilde{\boldsymbol{w}}\tau$ is an orderly circuit for S, with $S\boldsymbol{q}\tilde{\tau} = S\tilde{\tau}\boldsymbol{w} = W$. The example of Figure 5 shows that, with $\boldsymbol{q} = \boldsymbol{w} = \nu\zeta$, such a circuit need not be regular. □

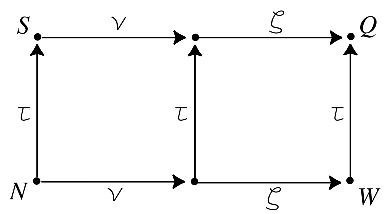

Figure 5: Under the hypotheses of Lemma 26, with $NS\,\mathfrak{L}\,WQ$, the orderly circuit $\boldsymbol{q}\tilde{\tau}\tilde{\boldsymbol{w}}\tau = \nu\zeta\tilde{\tau}\tilde{\nu}\tilde{\zeta}\tau$ for S is not regular. For example, $\zeta\tilde{\tau}\tilde{\zeta}$ is not concise for $S\nu$ (cf. Definition 12).

27 Convention. Any graph representing a medium comes implicitly equipped with its like relation $\mathfrak{L}$. When several such graphs are considered (say, for different media), their respective like relations are distinguished by diacritics, such as $\mathfrak{L}'$ or $\mathfrak{L}^*$.

28 Theorem. *Any graph representing a medium is mediatic.*

Proof. Because any graph isomorphic to a mediatic graph is mediatic, we can invoke Lemma 15 and content ourselves with proving that the graph of a medium is mediatic (which simplifies our notation). Denote the medium by $\mathcal{M} = (\mathcal{S}, \mathcal{T})$, and let $G = (\mathcal{S}, E)$ be its graph. We prove that G satisfies [G1], [G2] and [G3].

[G1] Axiom [Ma] requires that G be connected.

[G2] Axiom [Mb] implies that G must be even. Hence, by König's Theorem, it must be bipartite.

[G3] Suppose that $NS\,\mathfrak{L}\,PR\,\mathfrak{L}\,WQ$. By Lemma 26 (applied twice), there must be some tokens τ and μ such that $N\tau = S$, $P\tau = R$, $P\mu = R$ and $W\mu = Q$, so $\tau = \mu$. Let then $\boldsymbol{q}$ and $\boldsymbol{w}'$ be two concise messages from S, and let $\boldsymbol{w}$ and bq' be two concise messages from N, such that

$$S\boldsymbol{q} = Q, \quad S\boldsymbol{w}' = W, \quad N\boldsymbol{w} = W, \quad N\boldsymbol{q}' = Q.$$

The situation is exactly as in Theorem 11, with the same notation. Because $\tau = \mu$, Condition (ii) of this theorem holds. We conclude that Conditions (iii) and (iv) also hold, which leads to

$$\delta(S, Q) + 1 = \delta(N, W) + 1 = \delta(S, W) = \delta(N, Q).$$

We have thus $NS\,\mathfrak{L}\,WQ$; so Axiom [G3] holds. □

We omit the proof of the next lemma, which is straightforward.

29 Lemma. *Let $G = (V, E)$ and $G' = (V', E')$ be two mediatic graphs, with their respective like relations $\mathfrak{L}$ and $\mathfrak{L}'$, and let φ be a bijection of V onto V'. Then φ is an isomorphism of G onto G' if and only if*

$$ST\,\mathfrak{L}\,PQ \quad \Longleftrightarrow \quad \varphi(S)\varphi(T)\,\mathfrak{L}'\varphi(P)\varphi(Q) \qquad (S, T, P, Q \in V).$$

30 Remark. The like relation is the fundamental tool for the study of mediatic graphs. We shall see that any mediatic graph G can be used to construct a medium $\mathcal{M}$ that has G as its graph. Each of the equivalence classes $\langle ST\rangle$ of the like relation contains 'parallel' arcs of the graph, and will turn out to correspond to a particular token, say τ, of the medium under construction, with the class $\langle TS\rangle$ corresponding to the reverse token $\tilde{\tau}$. Before proceeding to such a construction, we establish in Theorem 32 a useful result which precisely links the isomorphism of media to that of their graphs.

31 Definition. Two media $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{S}', \mathcal{T}')$ are isomorphic if there exists a pair (α, β) of bijections $\alpha : \mathcal{S} \to \mathcal{S}'$ and $\beta : \mathcal{T} \to \mathcal{T}'$ such that

$$S\tau = V \Longleftrightarrow \alpha(S)\beta(\tau) = \alpha(V) \qquad (S, V \in \mathcal{S},\ \tau \in \mathcal{T}). \tag{11}$$

Paired Isomorphisms of Media and Graphs

Isomorphic media yield isomorphic mediatic graphs, and vice versa.

32 Theorem. *Suppose that $\mathcal{M} = (\mathcal{S}, \mathcal{T})$ and $\mathcal{M}' = (\mathcal{S}', \mathcal{T}')$ are two media and let $G = (\mathcal{S}, E)$ and $G' = (\mathcal{S}', E')$ be their respective graphs. Then $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic if and only if G and G' are isomorphic; more precisely:*

(i) *if (α, β) is an isomorphism of $\mathcal{M}$ onto $\mathcal{M}'$, then $\alpha : \mathcal{S} \to \mathcal{S}'$ is an isomorphism of G onto G' in the sense of* (9)*;*

(ii) *if $\varphi : \mathcal{S} \to \mathcal{S}'$ is an isomorphism of G onto G' in the sense of* (9)*, then there exists a bijection $\beta : \mathcal{T} \to \mathcal{T}'$ such that (φ, β) is an isomorphism of $\mathcal{M}$ onto $\mathcal{M}'$.*

Proof. (i) Suppose that (α, β) is an isomorphism of $\mathcal{M}$ onto $\mathcal{M}'$. For any two distinct S, T in $\mathcal{S}$, we have successively

$$\begin{aligned}
&\{S,T\} \in E \\
&\quad\Longleftrightarrow (\exists \tau \in \mathcal{T})(S\tau = T) && (G \text{ is the graph of } \mathcal{M}) \\
&\quad\Longleftrightarrow (\exists \tau \in \mathcal{T})(\alpha(S)\beta(\tau) = \alpha(T)) && (\mathcal{M} \text{ and } \mathcal{M}' \text{ are isomorphic}) \\
&\quad\Longleftrightarrow \{\alpha(S), \alpha(T)\} \in E' && (G' \text{ is the graph of } \mathcal{M}'),
\end{aligned}$$

and so

$$\{S,T\} \in E \iff \{\alpha(S), \alpha(T)\} \in E' \qquad (S,T \in \mathcal{S},\ S \neq T).$$

We conclude that $\alpha : \mathcal{S} \to \mathcal{S}'$ is an isomorphism of G onto G'.

(ii) Let $\varphi : \mathcal{S} \to \mathcal{S}'$ be an isomorphism of G onto G'. Define a function $\beta : \mathcal{T} \to \mathcal{T}'$ by

$$\beta(\tau) = \tau' \quad \iff \quad (\forall S, T \in \mathcal{S})(S\tau = T \Leftrightarrow \varphi(S)\tau' = \varphi(T)). \tag{12}$$

We first verify that the r.h.s. of the equivalence (12) correctly defines β as a bijection of $\mathcal{T}$ onto $\mathcal{T}'$. For any $\tau \in \mathcal{T}$, there exists distinct states S and T in $\mathcal{S}$ such that $S\tau = T$ and $\{S,T\} \in E$. Fix S and T temporarily. By the isomorphism $\varphi : \mathcal{S} \to \mathcal{S}'$ of G onto G', we have $\{\varphi(S), \varphi(T)\} \in E'$, and because G' is the graph of $\mathcal{M}'$, we necessarily have $\varphi(S)\tau' = \varphi(T)$ for some $\tau' \in \mathcal{T}'$, which is unique by Lemma 5(i). The hypothesis that φ is an isomorphism of G onto G' ensures that the r.h.s. of (12) is indeed an equivalence.

Next, we show that $\beta(\tau)$ does not depend upon the choice of S and T. Let P, Q be another pair of distinct states in $\mathcal{S}$ such that $P\tau = Q$, and let $P = S\boldsymbol{m}$ and $Q = T\boldsymbol{n}$ for some concise messages $\boldsymbol{m} = \tau_1 \dots \tau_m$ and $\boldsymbol{n} = \mu_1 \cdots \mu_n$. By Condition [M4], $\tau\boldsymbol{n}$ and $\boldsymbol{m}\tau$ are concise messages, and so Theorem 11 applies. Invoking its implication (ii) $\Rightarrow$ (iii), we get $\ell(\boldsymbol{m}) = \ell(\boldsymbol{n})$ and $\mathcal{C}(\boldsymbol{m}) = \mathcal{C}(\boldsymbol{n})$, yielding $m = n$. Denote by $\mathfrak{L}$ and $\mathfrak{L}'$ the like relations of G and G' respectively. We have thus shown that $ST \,\mathfrak{L}\, PQ$. By Lemma 29, we also have

$$\varphi(S)\varphi(T) \,\mathfrak{L}'\, \varphi(P)\varphi(Q).$$

Since we have $\varphi(S)\tau' = \varphi(T)$, we can apply Lemma 26 and derive $\varphi(P)\tau' = \varphi(Q)$.

We still have to prove that β is indeed a bijection. For any $\tau' \in \mathcal{T}'$ there are some $S', T' \in \mathcal{T}'$ such that $S'\tau' = T'$. We have thus $\{S', T'\} \in E'$, and since φ is an isomorphism of G onto G', also $\{\varphi^{-1}(S'), \varphi^{-1}(T')\} \in E$, with $\varphi^{-1}(S')\tau = \varphi^{-1}(T')$ for some $\tau \in \mathcal{T}$. Thus β maps $\mathcal{T}$ onto $\mathcal{T}'$. Suppose now that $\beta(\tau) = \beta(\mu) = \tau' \in \mathcal{T}'$. This implies that for some $S, T, P, Q \in \mathcal{S}$ and $N, M \in \mathcal{S}'$, we must have

$$S\tau = T, \quad P\mu = Q, \text{ and } N\tau' = M, \tag{13}$$

together with $\varphi(S) = \varphi(P) = N$ and $\varphi(T) = \varphi(Q) = M$ by the definition of β. As φ is a 1-1 function, we obtain $S = P$ and $T = Q$ in (13). Using Lemma 5(ii), we get $\tau = \mu$. Thus, β is a 1-1 function and so a bijection.

The fact that (φ, β) is an isomorphism of $\mathcal{M}$ onto $\mathcal{M}'$ follows from the definition of β by (12). We have

$$S\tau = T \iff \varphi(S)\beta(\tau) = \varphi(T) \qquad (S, T \in \mathcal{S})$$

whether or not $\{S,T\} \in E$. $\square$

Having defined the graph of a medium and shown that such a graph was necessarily mediatic, we now go in the opposite direction and construct a medium from an arbitrary mediatic graph.

From Mediatic Graphs to Media

33 Definition. Let $G = (\mathcal{S}, E)$ be a mediatic graph and let $\mathfrak{L}$ be its like relation. For any $ST \in \vec{E}$, define a transformation $\tau_{ST} : \mathcal{S} \to \mathcal{S} \,:\, P \mapsto P\tau_{ST}$ by the formula

$$P\tau_{ST} = \begin{cases} Q \text{ if } ST \,\mathfrak{L}\, PQ, \\ P \text{ otherwise.} \end{cases} \tag{14}$$

We denote by $\mathcal{T} = \{\tau_{ST} \mid ST \in \vec{E}\}$ the set containing all those transformations. It is clear that the pair $(\mathcal{S}, \mathcal{T})$ is a token system. Such a token system is said to be *induced* by the mediatic graph G. The theorem below establishes that a token system $\mathcal{K}$ induced by a mediatic graph G is in fact a medium. We say that $\mathcal{K}$ is the *medium of the graph* G. Notice that, since $\mathfrak{L}$ is an equivalence relation on $\vec{E}$, we have $\tau_{ST} = \tau_{PQ}$ whenever $ST \,\mathfrak{L}\, PQ$. In such a case, we have in fact $\langle ST \rangle = \langle PQ \rangle$. The choice of a particular pair $ST \in \langle PQ \rangle$ to denote a token τ_{ST} is thus arbitrary. Notice that, as a consequence of this definition, whenever $\{S, T\} \in E$, then also $ST \,\mathfrak{L}\, ST$, and so $S\tau_{ST} = T$.

This construction is motivated by the following theorem.

34 Theorem. *The token system $(\mathcal{S}, \mathcal{T})$ induced by a mediatic graph $G = (\mathcal{S}, E)$ is a medium. In particular, the tokens τ_{ST} and τ_{TS} defined by* (14) *are mutual reverses for any $\{S, T\} \in E$.*

Proof. We verify that $(\mathcal{S}, \mathcal{T})$ satisfies Axioms [Ma] and [Mb] of a medium.

[Ma] For any $S, T \in \mathcal{S}$, there is a shortest path $S_0 = S, S_1, \ldots, S_n = T$ between S and T in G. This implies that, for $0 \leq i \leq n-1$, we have $\{S_i, S_{i+1}\} \in E$, which yields $S_i\tau_{S_iS_{i+1}} = S_{i+1}$. It follows that the message $\boldsymbol{m} = \tau_{S_0S_1} \ldots \tau_{S_{n-1}S_n}$ produces T from S and is stepwise effective. To prove that $\boldsymbol{m}$ is concise, we must still show that it is consistent and without repetitions. The message $\boldsymbol{m}$ is consistent since otherwise we would have

$$S_h\tau_{MN} = S_{h+1} \quad \text{and} \quad S_k\tau_{NM} = S_{k+1} \tag{15}$$

for some indices h and k, with $h < k$, and some $NM \in \vec{E}$. Since τ_{MN} is the reverse of τ_{NM}, the last equality in (15) can be rewritten as $S_{k+1}\tau_{MN} = S_k$. Thus, by definition of the tokens in (14), the above statement (15) leads to $S_hS_{h+1} \,\mathfrak{L}\, MN \,\mathfrak{L}\, S_{k+1}S_k$ which, by transitivity, gives $S_kS_{k+1} \,\mathfrak{L}\, S_{h+1}S_h$. Because $h < k$, we derive by the definition of the like relation $\mathfrak{L}$

$$k + 1 - h = \delta(S_{k+1}, S_h) = \delta(S_k, S_{h+1}) = k - 1 - h$$

yielding the absurdity $1 = -1$. Thus, $\boldsymbol{m}$ is consistent. Suppose that $\boldsymbol{m}$ has repeated tokens, say $S_i\tau_{S_iS_{i+1}} = S_{i+1}$ and $S_{i+k}\tau_{S_iS_{i+1}} = S_{i+k+1}$ for some indices $0 \leq i < n$ and $0 \leq i + k < n$. This would give $S_iS_{i+1} \,\mathfrak{L}\, S_{i+k}S_{i+k+1}$, leading to

$$d(S_i, S_{i+k+1}) = k + 1 > k - 1 = d(S_{i+1}, S_{i+k}),$$

while by the definition of $\mathfrak{L}$ we should have $d(S_i, S_{i+k+1}) = d(S_{i+1}, S_{i+k})$, a contradiction. Thus, the message $\boldsymbol{m}$ is concise.

[Mb] Let $\boldsymbol{m} = \tau_{S_0S_1}\tau_{S_1S_2} \ldots \tau_{S_{n-1}S_n}$ be a return message for some state S; we have thus $S_0 = S_n = S$. In the terminology of G, we have a closed walk $S = S_0, S_1, \ldots, S_n = S$. We denote this closed walk by $\mathbf{W}$ and we write $\vec{E}_{\mathbf{W}}$ for the set of all its arcs S_iS_{i+1},

$0 \leq i \leq n-1$. By [G2] and König's Theorem, such a closed walk is even; so $n = 2q$ for some $q \in \mathbb{N}$. We prove by induction on q that $\boldsymbol{m}$ is vacuous. The case $q = 1$ (the smallest possible return) is trivial, so we suppose that [Mb] holds for any $1 \leq p < q$ and prove that [Mb] also holds for $q = p$. We consider two cases.

Case 1: $\mathbf{W}$ is an isometric subgraph of G. Thus, $\mathbf{W}$ is a minimal circuit of G. Take any token $\tau_{S_iS_{i+1}}$ in $\boldsymbol{m}$. Since (with the addition modulo k in the indices), we have for $0 \leq i < n$

$$\begin{aligned} \delta(S_{i+1}, S_{i+k}) &= \delta(S_i, S_{i+k+1}) = k-1, \\ \delta(S_i, S_{i+k}) &= \delta(S_{i+1}, S_{i+k+1}) = k, \end{aligned}$$

we obtain $S_iS_{i+1} \,\mathfrak{L}\, S_{i+k+1}S_{i+k}$. By the definition of the tokens in (14) and the transitivity and symmetry of $\mathfrak{L}$, we get for any $P, Q \in \mathcal{S}$

$$\begin{aligned} P\tau_{S_iS_{i+1}} = Q &\Longleftrightarrow S_iS_{i+1} \,\mathfrak{L}\, PQ \\ &\Longleftrightarrow S_{i+k+1}S_{i+k} \,\mathfrak{L}\, PQ \\ &\Longleftrightarrow P\tau_{S_{i+k+1}S_{i+k}} = Q \\ &\Longleftrightarrow Q\tau_{S_{i+k}S_{i+k+1}} = P. \end{aligned}$$

We conclude that $\tau_{S_{i+k}S_{i+k+1}}$ and $\tau_{S_iS_{i+1}}$ are mutual reverses, and so $\boldsymbol{m}$ is vacuous. (Note that the induction hypothesis has not been used here.)

Case 2: $\mathbf{W}$ is not an isometric subgraph of G. Then, there are two vertices S_i and S_j in $\mathbf{W}$, with $i < j$, and a shortest path $\mathbf{L}$ from S_i to S_j in G with $\delta_{ij} = \delta(S_i, S_j) < \min\{j-i, i+n-j\}$ (see Figure 6). Thus, $j - i$ and $i + n - j$ are the lengths of the two segments of $\mathbf{W}$ with endpoints S_i and S_j. For simplicity, we can assume without loss of generality that S_i and S_j are the only vertices of $\mathbf{L}$ that are also in $\mathbf{W}$. Let $\boldsymbol{p}$ the straight message producing S_j from S_i and corresponding to the shortest path $\mathbf{L}$ in the sense of Lemma 16.

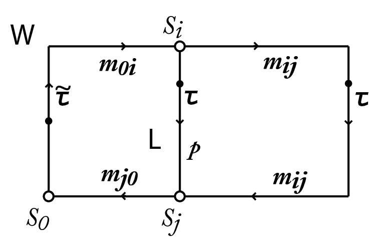

Figure 6: Case 2 in the proof of Axiom [Mb] in Theorem 34: the closed walk $\mathbf{W}$ is not an isometric subgraph.

We also split $\boldsymbol{m}$ into the three messages:

$$\begin{aligned} \boldsymbol{m}_{0i} &= \tau_{S_0S_1} \ldots \tau_{S_{i-1}S_i} \\ \boldsymbol{m}_{ij} &= \tau_{S_iS_{i+1}} \ldots \tau_{S_{j-1}S_j} \\ \boldsymbol{m}_{j0} &= \tau_{S_jS_{j+1}} \ldots \tau_{S_{n-1}S_0}\,. \end{aligned}$$

We have thus $\boldsymbol{m} = \boldsymbol{m}_{0i}\boldsymbol{m}_{ij}\boldsymbol{m}_{j0}$. Note that the two messages $\boldsymbol{m}_{0i}\boldsymbol{p}\boldsymbol{m}_{j0}$ and $\widetilde{\boldsymbol{p}}\boldsymbol{m}_{ij}$ have a length strictly smaller that $n = 2q$. By the induction hypothesis, these two messages are

vacuous. Accordingly, for any token τ of $\boldsymbol{p}$, there is an reverse token $\tilde{\tau}$ either in $\boldsymbol{m}_{0i}$ or in $\boldsymbol{m}_{j0}$. (In Figure 6 the token $\tilde{\tau}$ is pictured as being part of $\boldsymbol{m}_{0i}$.) Considered from the viewpoint of the message $\widetilde{\boldsymbol{p}}\boldsymbol{m}_{ij}$ from S_j, the token $\tilde{\tau}$ is in $\widetilde{\boldsymbol{p}}$ with its reverse τ in $\boldsymbol{m}_{ij}$. The two reverses of the tokens in $\boldsymbol{p}$ and $\widetilde{\boldsymbol{p}}$, form a pair of mutually reverse tokens $\{\tau, \tilde{\tau}\}$ in $\boldsymbol{m}$. Such a pair can be obtained for any token τ in $\boldsymbol{p}$. Augmenting the set of all those pairs by the set of mutually reverse tokens in $\boldsymbol{m}_{0i}$, $\boldsymbol{m}_{ij}$ and $\boldsymbol{m}_{j0}$, we obtain a partition of the set $\mathcal{C}(\boldsymbol{m})$ into pairs of mutually reverse tokens, which establishes that the message $\boldsymbol{m}$ is vacuous.

We have shown that the token system $(\mathcal{S}, \mathcal{T})$ satisfies Axioms [Ma] and [Mb]. The proof is thus complete. □

35 Remark. In the above proof, the inductive argument used to establish Case 2 of [M3] may convey the mistaken impression that the situation is always straightforward. The simple graph pictured in Figure 6 is actually glossing over some intricacies. The non–isometric subgraph **W** is pictured by the thick lines in Figure 7 and is not 'convex.' We can see how the inductive stage splitting the closed walk **W** by the shortest path **L** may lead to form, in each of the two smaller closed walks, pairs $\{\mu, \tilde{\mu}\}$ and $\{\nu, \tilde{\nu}\}$ which correspond in fact to the same pair of tokens in **W**. Since the arcs corresponding to μ and ν are in the like relation $\mathfrak{L}$, the mistaken assignment is temporary.

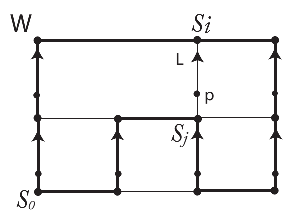

Figure 7: The non–isometric subgraph **W** of Case 2 in the proof of [M3] in Theorem 34 is pictured in thick lines. The inductive stage of the proof leads to form temporarily, in each of the two smaller closed walks delimited by the shortest path **L**, pairs $\{\mu, \tilde{\mu}\}$ and $\{\nu, \tilde{\nu}\}$ corresponding to the same pair of mutually reverse tokens in **W**.

We finally obtain:

36 Theorem. *Let $\mathcal{S}$ an arbitrary set, with $|\mathcal{S}| \geq 2$. Denote by $\mathfrak{M}$ the set of all media on $\mathcal{S}$, and by $\mathfrak{G}$ the set of all mediatic graphs on $\mathcal{S}$. There exists a bijection $\mathfrak{f} : \mathfrak{M} \to \mathfrak{G} : \mathcal{M} \mapsto \mathfrak{f}(\mathcal{M})$ such that $G = \mathfrak{f}(\mathcal{M})$ is the graph of $\mathcal{M}$ in the sense of Definition* 14 *if and only if $\mathcal{M}$ is the medium of the mediatic graph G in the sense of Definition* 33.

Proof. Because the set $\mathcal{S}$ of states is constant in $\mathfrak{M}$ and confounded with the constant set of vertices in $\mathfrak{G}$, we could reinterpret the function $\mathfrak{f}$ as a mapping of the family $\mathfrak{T}$ of all sets of token $\mathcal{T}$ making $(\mathcal{S}, \mathcal{T})$ a medium, into the family $\mathfrak{E}$ of all sets of edges E making $(\mathcal{S}, E)$ a mediatic graph. However, any set of edges E of a mediatic graph on $\mathcal{S}$ is characterized by its like relation $\mathfrak{L}$, or equivalently, by the partition of $\vec{E}$ induced by $\mathfrak{L}$. We choose the latter characterization for the purpose of this proof, and denote by $\vec{\mathfrak{E}}_{|lr}$ the set of all the partitions of the sets of arcs $\vec{E}$ induced by the like relations characterizing the sets of edges in the collection $\mathfrak{E}$.

From Lemmas 28 and 34, we know that the graph of a medium is mediatic, and that the token system induced by a mediatic graph is a medium. We have to show that the functions

$$\mathfrak{f} : \mathfrak{T} \to \vec{\mathfrak{E}}_{|lr} \quad \text{and} \quad \mathfrak{g} : \vec{\mathfrak{E}}_{|lr} \to \mathfrak{T}$$

implicitly defined by (8) and (14), respectively, are mutual inverses. Note that, for any $\mathcal{T} \in \mathfrak{T}$, the partition $\mathfrak{f}(\mathcal{T})$ is defined via a function f mapping $\mathcal{T}$ into the partition $\mathfrak{f}(\mathcal{T})$. Writing as before $\langle ST \rangle$ for the equivalence class containing the arc ST, we have

$$P\tau = Q \quad \Longleftrightarrow \quad f(\tau) = \langle PQ \rangle \qquad (\tau \in \mathcal{T};\, P, Q \in \mathcal{S}). \tag{16}$$

Proceeding similarly, but inversely, for the function $\mathfrak{g}$, we notice that it defines, for each $\vec{E}_{|lr}$ in $\vec{\mathfrak{E}}_{|lr}$ the set of tokens $\mathfrak{g}(\vec{E}_{|lr})$ via a function g mapping $\vec{E}_{|lr}$ into the set of tokens $\mathfrak{g}(\vec{E}_{|lr})$; we obtain

$$\langle ST \rangle = \langle PQ \rangle \quad \Longleftrightarrow \quad Pg(\langle ST \rangle) = Q \qquad (S, T, P, Q \in \mathcal{S}). \tag{17}$$

Combining (16) and (17) we obtain

$$P\tau = Q \Longleftrightarrow f(\tau) = \langle PQ \rangle \Longleftrightarrow P(g \circ f)(\tau) = Q \quad (\tau \in \mathcal{T};\, P, Q \in \mathcal{S}).$$

We have thus $g = f^{-1}$ and so $\mathfrak{g} = \mathfrak{f}^{-1}$. Conversely, we have

$$\langle ST \rangle = \langle PQ \rangle \Longleftrightarrow Pg(\langle ST \rangle) = Q \Longleftrightarrow (f \circ g)(\langle ST \rangle) = \langle PQ \rangle \qquad (S, T, P, Q \in \mathcal{S}),$$

yielding $f = g^{-1}$ and so $\mathfrak{f} = \mathfrak{g}^{-1}$. □

37 Two Examples. In the last paragraph of our introductory section, we announced that the collection $\mathfrak{I}$ of all the interval orders on a finite set X was representable as a mediatic graph. The argument goes as follows. Doignon and Falmagne (1997) proved that such a collection $\mathfrak{I}$ is always 'well-graded', that is, for any two interval orders K and L, there exists a sequence $K_0 = K, K_1, \ldots, K_n = L$ of interval orders on X such that $|K_i \triangle K_{i+1}| = 1$ for $0 \leq i \leq n-1$ and $|K \triangle L| = n$. It is easily shown (see Falmagne, 1997) that any well-graded family $\mathcal{F}$ can be cast as a medium $\mathcal{M}(\mathcal{F})$: the states of the medium are the sets of the family, and the tokens consist in either adding or removing an element from a set in $\mathcal{F}$. By Theorem 28, the graph of the medium $\mathcal{M}(\mathcal{I})$ is mediatic. A similar argument applies to the family of all the semiorders on X, and to some other families on X (for example, partial orders and biorders, cf. Doignon and Falmagne, 1997).

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